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Eigenfunctions of the Laplacian Acting on Degree Zero Bundles over Special Riemann Surfaces

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ABSTRACT

We find an infinite set of eigenfunctions for the Laplacian with respect to a flat metric with conical singularities and acting on degree zero bundles over special Riemann surfaces of genus greater than one. These special surfaces correspond to Riemann period matrices satisfying a set of equations which lead to a number theoretical problem. It turns out that these surfaces precisely correspond to branched covering of the torus. This reflects in a Jacobian with a particular kind of complex multiplication.

1 Introduction

In [1] it was considered the problem of constructing a set of eigenfunctions of the Laplacian acting on degree zero bundles over Riemann surfaces. The corresponding metric is given by the modulo square of a particular holomorphic one-differential $\omega_{n,m}$. In [1] it has been also shown that eigenvalues with a nontrivial dependence on the complex structure may be obtained as solutions of the equation

$$\omega_{n',m'} = c\,\omega_{n,m}. \tag{1.1}$$

This equation is equivalent to

$$m'_j - \sum_{k=1}^{h} \Omega_{jk} n'_k = \overline{c}\left(m_j - \sum_{k=1}^{h} \Omega_{jk} n_k\right), \qquad j = 1, \ldots, h, \tag{1.2}$$

where m_j, n_j, m'_j, n'_j are integers. In this paper we find a set of solutions of such an equation. The general problem involved in Eq.(1.1) concerns the properties of the Riemann period matrix and its number theoretical structure. We will call special Riemann surfaces those with a Riemann period matrix satisfying Eq.(1.2).

2 Primitive differentials and scalar products

In this section, after fixing the notation, we introduce an infinite set of holomorphic one-differentials (we shall call them *primitive differentials*) which can be considered as the building blocks for our investigation. These differentials have been previously introduced in [1]. Next, we consider scalar products defined in terms of monodromy factors associated to the primitive differentials. By Riemann bilinear relations we generalize a result in [1] obtaining a relation between scalar product, monodromy factors and surface integrals. These aspects are reminiscent of the well-known relations between area, Fuchsian dilatations and Laplacian spectra which arise for example in the framework of the Selberg trace formula.

2.1 Notation and definitions

Let Σ be a compact Riemann surface of genus $h \geq 1$ and $\alpha_1, \ldots, \alpha_h, \beta_1, \ldots \beta_h$ a symplectic basis of the first homology group $H_1(\Sigma, \mathbb{Z})$, that is with intersection matrix

$$\begin{pmatrix} \alpha\cdot\alpha & \alpha\cdot\beta \\ \beta\cdot\alpha & \beta\cdot\beta \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \tag{2.1}$$

where $\mathbb{I}$ is the $h \times h$ identity matrix. Let $\omega_1, \ldots, \omega_h$ be the basis of the $\mathbb{C}$ vector space of holomorphic one-differentials with the standard normalization

$$\oint_{\alpha_j} \omega_k = \delta_{jk}. \tag{2.2}$$

The Riemann period matrix is defined by

$$\Omega_{jk} \doteq \oint_{\beta_j} \omega_k. \tag{2.3}$$

By means of the Riemann bilinear relations [2]

$$\int_\Sigma \omega_j \wedge \overline{\omega}_k = \sum_{l=1}^h \left(\oint_{\alpha_l} \omega_j \oint_{\beta_l} \overline{\omega}_k - \oint_{\alpha_l} \overline{\omega}_k \oint_{\beta_l} \omega_j \right), \tag{2.4}$$

it follows that $\Omega_{ij} = \Omega_{ji}$, and $\det \Omega^{(2)} > 0$, where $\Omega^{(1)}_{kj} \doteq \Re\,\Omega_{kj}$ and $\Omega^{(2)}_{kj} \doteq \Im\,\Omega_{kj}$. We shall denote elements in $H_1(\Sigma, \mathbb{Z})$ by

$$\gamma_{p,q} \doteq p \cdot \alpha + q \cdot \beta, \qquad (q,p) \in \mathbb{Z}^{2h}, \tag{2.5}$$

in particular

$$\oint_{\gamma_{p,q}} \omega \doteq \sum_{k=1}^h \left(p_k \oint_{\alpha_k} \omega + q_k \oint_{\beta_k} \omega \right), \tag{2.6}$$

and

$$\int_{z_0}^{z+\gamma_{p,q}} \omega \doteq \int_{z_0}^z \omega + \oint_{\gamma_{p,q}} \omega, \tag{2.7}$$

where, $z_0 \in \Sigma$, $z \in \Sigma$ and ω is an arbitrary meromorphic one-differential.

2.2 Primitive differentials

Let us consider the set $\mathcal{K}(\Sigma) \subset H^1(\Sigma)$ of all nonzero real, harmonic one-forms on Σ which are integral, *i.e.* such that

$$\oint_\gamma \alpha \in \mathbb{Z}, \qquad \gamma \in H_1(\Sigma, \mathbb{Z}). \tag{2.8}$$

Note that $\mathcal{K}(\Sigma)$ is a lattice in the $2h$ real vector space $H^1(\Sigma)$. Let us consider the set of holomorphic one-differentials

$$\mathcal{H}(\Sigma) \doteq \{\omega = \pi i(\alpha + i^\star\alpha) | \alpha \in \mathcal{K}(\Sigma)\}, \tag{2.9}$$

where * denotes the conjugation operator whose action on a one-form $\eta = u(z)dz + v(z)d\bar{z}$ is $^\star\eta = -iu(z)dz + iv(z)d\bar{z}$. Note that $\alpha = \Im\,\omega/\pi$ and $^\star\alpha = -i\Re\,\omega/\pi$.

The elements of $\mathcal{H}(\Sigma)$ correspond to the following *primitive differentials* [1]

$$\omega_{n,m} \doteq \sum_{k=1}^{h} c_{n,m;k}\omega_k, \tag{2.10}$$

where

$$c_{n,m;k} = \pi \sum_{j=1}^{h} \left(m_j - \sum_{l=1}^{h} n_l \overline{\Omega}_{lj}\right) \left({\Omega^{(2)}}^{-1}\right)_{jk}, \qquad (n,m) \in \mathbb{Z}^{2h}, \tag{2.11}$$

$k = 1, \ldots, h$. Let us set

$$f_{n,m}(z) \doteq e^{\int_{z_0}^{z} \omega_{n,m}}, \tag{2.12}$$

where z_0 is fixed on Σ and $z \in \Sigma$. Note that the monodromy of $f_{n,m}$ takes real values, that is

$$(n,m|q,p) \in \mathbb{R}, \qquad (q,p) \in \mathbb{Z}^{2h}. \tag{2.13}$$

where

$$(n,m|q,p) \doteq e^{\oint_{\gamma_{p,q}} \omega_{n,m}} = \frac{f_{n,m}(z+\gamma_{p,q})}{f_{n,m}(z)} = \exp\left[\sum_{j=1}^{h} (p_j + \sum_{k=1}^{h} q_k \Omega_{kj}) c_{n,m;j}\right]. \tag{2.14}$$

For later use we define the coefficients

$$D_{kj}^{nm} \doteq m_k \delta_{kj} - n_k \overline{\Omega}_{kj}, \tag{2.15}$$

so that

$$c_{n,m;k} = \pi \sum_{j,l=1}^{h} D_{jl}^{nm} ({\Omega^{(2)}}^{-1})_{lk}. \tag{2.16}$$

2.3 Monodromy and scalar products

Let us define the scalar product

$$\langle\langle n,m|q,p\rangle\rangle \doteq \oint_{\gamma_{p,q}} \omega_{n,m}, \qquad (n,m;q,p) \in \mathbb{Z}^{4h}. \tag{2.17}$$

By (2.11) we have

$$\langle\langle n,m|q,p\rangle\rangle = \pi \sum_{j,k=1}^{h} \left(p_j + \sum_{i=1}^{h} q_i \Omega_{ij}\right) \left({\Omega^{(2)}}^{-1}\right)_{jk} \left(m_k - \sum_{l=1}^{h} \overline{\Omega}_{kl} n_l\right), \tag{2.18}$$

which has the properties

$$\overline{\langle\langle n,m|q,p\rangle\rangle} = \langle\langle -q,p|-n,m\rangle\rangle = \langle\langle n,m|q,p\rangle\rangle - 2i\pi(p\cdot n + q\cdot m), \tag{2.19}$$

so that

$$\Im\langle\langle n,m|q,p\rangle\rangle=\Im\langle\langle m,n|p,q\rangle\rangle. \tag{2.20}$$

Furthermore

$$\langle\langle n,m|q,p\rangle\rangle=\frac{1}{2\pi i}\sum_{j=1}^{h}\left(\langle\langle n,m|\hat{\jmath},0\rangle\rangle\langle\langle 0,\hat{\jmath}|q,p\rangle\rangle+\langle\langle n,m|0,\hat{\jmath}\rangle\rangle\langle\langle\hat{\jmath},0|q,p\rangle\rangle\right), \tag{2.21}$$

where $\hat{\jmath}\doteq(\delta_{1j},\delta_{2j},\ldots,\delta_{hj})$. By (2.13) and (2.19) it follows that

$$(n,m|q,p)=e^{\langle\langle n,m|q,p\rangle\rangle}=e^{\overline{\langle\langle n,m|q,p\rangle\rangle}}=(-q,p|-n,m). \tag{2.22}$$

Note that by (2.19) the scalar product

$$\langle n,m|q,p\rangle\doteq\Re\oint_{\gamma_{p,-q}}\omega_{n,m}=\Re\langle\langle n,m|-q,p\rangle\rangle=-\Re\langle\langle n,m|q,-p\rangle\rangle, \tag{2.23}$$

has the symmetry property

$$\langle n,m|q,p\rangle=\langle q,p|n,m\rangle, \tag{2.24}$$

and

$$\langle n,m|q,p\rangle=\langle\langle n,m|-q,p\rangle\rangle-i\pi(p\cdot n-q\cdot m). \tag{2.25}$$

Furthermore, by (2.18) and (2.19)

$$\langle n,m|q,p\rangle=\pi\sum_{j,k=1}^{h}\left[\left(p_j-\sum_{i=1}^{h}q_i\Omega^{(1)}_{ij}\right)\left({\Omega^{(2)}}^{-1}\right)_{jk}\left(m_k-\sum_{l=1}^{h}\Omega^{(1)}_{kl}n_l\right)+q_j\Omega^{(2)}_{jk}n_k\right]. \tag{2.26}$$

Positivity of $\Omega^{(2)}$ implies that

$$\langle n,m|n,m\rangle\geq 0, \tag{2.27}$$

and $\langle n,m|n,m\rangle=0$ iff $|n,m\rangle=|0,0\rangle$. Observe that by (2.25) it follows that

$$\langle n,m|n,m\rangle=\langle\langle n,m|-n,m\rangle\rangle. \tag{2.28}$$

$\langle n,m|q,p\rangle$ can be expressed in terms of the coefficients $c_{n,m;k}$ only

$$\begin{aligned}\langle n,m|q,p\rangle &= \pi^{-1}\sum_{j,k=1}^{h}\left(a_{n,m;j}\Omega^{(2)}_{jk}a_{q,p;k}+b_{q,p;j}\Omega^{(2)}_{jk}b_{n,m;k}\right)\\ &= \pi^{-1}\sum_{j,k=1}^{h}\left(\langle n,m|0,\hat{\jmath}\rangle\Omega^{(2)}_{jk}\langle 0,\hat{k}|q,p\rangle+q_j\Omega^{(2)}_{jk}n_k\right),\end{aligned} \tag{2.29}$$

where $a_{n,m;k} \doteq \Re\, c_{n,m;k}$ and $b_{n,m;k} \doteq \Im\, c_{n,m;k}$, that is

$$a_{n,m;k} = \pi \sum_{j=1}^{h} \left(m_j - \sum_{l=1}^{h} n_l \Omega_{lj}^{(1)} \right) \left({\Omega^{(2)}}^{-1} \right)_{jk}, \qquad b_{n,m;k} = \pi n_k, \tag{2.30}$$

$k = 1, \ldots, h$, $(n, m) \in \mathbb{Z}^{2h}$. Finally, we note that in deriving Eq.(2.29) we used the relation $\omega_{n,m} = \sum_{k=1}^{h} \left(\oint_{\alpha_k} \omega_{n,m} \right) \omega_k$, that is

$$c_{n,m;k} = \oint_{\alpha_k} \omega_{n,m} = \langle\langle n, m|0, \hat{k}\rangle\rangle, \qquad a_{n,m;k} = \langle n, m|0, \hat{k}\rangle. \tag{2.31}$$

2.4 Duality, surface integrals and monodromy

Let us set

$$\rho_{n,m}^{(1)} \doteq \Re \sum_{k=1}^{h} d_{n,m;k}^{(1)} \omega_k. \tag{2.32}$$

We consider the problem of finding the structure of the coefficients $d_{n,m;k}^{(1)}$ such that

$$\oint_{\gamma_{p,q}} \rho_{n,m}^{(1)} = \oint_{\gamma_{m,n}} \rho_{q,p}^{(1)}, \tag{2.33}$$

$(n, m; q, p) \in \mathbb{Z}^{4h}$. This equation implies that

$$d_{n,m;k}^{(1)} = \sum_{j=1}^{h} \left(m_j + \sum_{l=1}^{h} n_l \Omega_{lj}^{(1)} \right) E_{jk} - \sum_{j=1}^{h} \left(i m_j + \sum_{l=1}^{h} n_l \Omega_{lj}^{(2)} \right) F_{jk} + i \sum_{l=1}^{h} n_l G_{lk}, \tag{2.34}$$

with E_{ij}, F_{ij}, G_{ij} real symmetric matrices not depending on (n, m). Note that the condition $\overline{d_{n,m;k}^{(1)}} = d_{n,m;k}^{(1)} - 2\pi i n_k$ is equivalent to $F_{jk} = 0, G_{jk} = \pi \delta_{jk}$. We now set

$$\rho_{n,m}^{(2)} \doteq \Re \sum_{k=1}^{h} d_{n,m;k}^{(2)} \omega_k, \tag{2.35}$$

and consider the condition

$$\oint_{\gamma_{p,-q}} \rho_{n,m}^{(2)} = \oint_{\gamma_{m,-n}} \rho_{q,p}^{(2)}, \tag{2.36}$$

that is

$$d_{n,m;k}^{(2)} = \sum_{j=1}^{h} \left(m_j - \sum_{l=1}^{h} n_l \Omega_{lj}^{(1)} \right) E_{jk} + \sum_{j=1}^{h} \left(i m_j + \sum_{l=1}^{h} n_l \Omega_{lj}^{(2)} \right) F_{jk} + i \sum_{l=1}^{h} n_l G_{lk} = \overline{d_{-n,m;k}^{(1)}}. \tag{2.37}$$

In this case the conditions

$$\overline{d_{n,m;k}^{(2)}} = d_{n,m;k}^{(2)} - 2\pi i n_k, \qquad E_{jk} = \pi \left({\Omega^{(2)}}^{-1} \right)_{jk}, \tag{2.38}$$

give $d^{(2)}_{n,m;k} = c_{n,m;k}$, that is

$$\rho^{(2)}_{n,m} = \Re\, \omega_{n,m}. \tag{2.39}$$

Thus $c_{n,m;k}$ can be fixed by imposing either the singlevaluedness of $\exp \Im \int^z \omega_{n,m}$ Eq.(2.13), or the duality condition (2.36), the same of Eq.(2.24) satisfied by $\Re\, \omega_{n,m}$, together with (2.38).

We will see that positivity and symmetry of $\langle \cdot | \cdot \rangle$ are at the basis of the fact that the monodromy of $f_{n,m}$ under a shift of z around $\gamma_{m,-n}$, is proportional to the area of the metric $|\omega_{n,m}|^2$. This metric defines a Laplacian of which $(f_{n,m}/\overline{f}_{n,m})^k$, $k \in \mathbb{Z}$, are eigenfunctions. These aspects are reminiscent of the well-known relationships between hyperbolic dilatations and eigenvalues of the Poincaré Laplacian.

By the Riemann bilinear relations we have

$$\begin{aligned} \int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{q,p} &= \sum_{j=1}^{h} \Big(\langle\langle n,m|0,\hat{\jmath}\rangle\rangle \overline{\langle\langle q,p|\hat{\jmath},0\rangle\rangle} - \overline{\langle\langle q,p|0,\hat{\jmath}\rangle\rangle} \langle\langle n,m|\hat{\jmath},0\rangle\rangle \Big) \\ &= \sum_{j=1}^{h} \left(\langle\langle n,m|0,\hat{\jmath}\rangle\rangle \langle\langle -\hat{\jmath},0|-q,p\rangle\rangle - \langle\langle n,m|\hat{\jmath},0\rangle\rangle \langle\langle 0,\hat{\jmath}|-q,p\rangle\rangle \right), \end{aligned} \tag{2.40}$$

where (2.19) has been used. By (2.21) and (2.25) we obtain

$$\frac{i}{2} \int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{q,p} = \pi \langle\langle n,m|-q,p\rangle\rangle = \pi \langle n,m|q,p\rangle + i\pi^2 (p\cdot n - q\cdot m), \tag{2.41}$$

so that the monodromy of $f_{n,m}$ corresponds to a surface integral, that is

$$f_{n,m}(z+\gamma_{p,-q}) = e^{\frac{1}{2\pi i}\int_\Sigma \overline{\omega}_{q,p}\wedge\omega_{n,m}} f_{n,m}(z). \tag{2.42}$$

By (2.19)(2.20) and (2.40) the surface integrals $\int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{q,p}$ have the properties

$$\int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{q,p} = \int_\Sigma \omega_{q,p} \wedge \overline{\omega}_{n,m} + 4\pi^2 (p\cdot n - q\cdot m), \tag{2.43}$$

and

$$\Im \frac{i}{2} \int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{q,p} = \Im \frac{i}{2} \int_\Sigma \omega_{m,n} \wedge \overline{\omega}_{p,q}. \tag{2.44}$$

The above structure suggests introducing the holomorphic one-differentials

$$\eta^{(1)}_j(z) = \pi \sum_{k=1}^{h} \left({\Omega^{(2)}}^{-1} \right)_{jk} \omega_k(z), \qquad j=1,\ldots,h, \tag{2.45}$$

$$\eta^{(2)}_j(z) = \pi \sum_{l=1}^{h} \left[i\delta_{jl} - \sum_{k=1}^{h} \Omega^{(1)}_{jk} \left({\Omega^{(2)}}^{-1} \right)_{kl} \right] \omega_l(z), \qquad j=1,\ldots,h. \tag{2.46}$$

By (2.10) and (2.11) it follows that

$$\omega_{n,m} = \sum_{k=1}^{h} (m_k \eta_k^{(1)} + n_k \eta_k^{(2)}), \tag{2.47}$$

moreover

$$\Im \oint_{\alpha_k} \eta_j^{(1)} = 0, \qquad \Im \oint_{\beta_k} \eta_j^{(1)} = \pi\delta_{jk}, \tag{2.48}$$

$$\Im \oint_{\alpha_k} \eta_j^{(2)} = \pi\delta_{jk}, \qquad \Im \oint_{\beta_k} \eta_j^{(2)} = 0, \tag{2.49}$$

$j, k = 1, \ldots, h$. Let us set

$$g_k^{(j)}(z) \doteq \exp \int_{z_0}^{z} \eta_k^{(j)}, \qquad j = 1, 2, \tag{2.50}$$

$k = 1, \ldots, h$. The expression of $f_{n,m}$ in terms of $g_k^{(j)}$ has the simple form

$$f_{n,m} = \prod_{k=1}^{h} g_k^{(1)^{m_k}} g_k^{(2)^{n_k}}. \tag{2.51}$$

Furthermore, since $\eta_k^{(2)} = -\sum_{j=1}^{h} \Omega_{kj}\eta_j$, it follows that

$$f_{n,m}(z) = \prod_{j,k=1}^{h} g_j^{(2)}(z)^{D_{kj}^{nm}} = e^{\sum_{j,k=1}^{h} D_{kj}^{nm} \int_{z_0}^{z} \eta_j}, \tag{2.52}$$

where the coefficients D_{kj}^{nm} have been defined in (2.15).

3 Eigenfunctions

Let α be an element of $\mathcal{K}(\Sigma)$ and $\omega = \pi i(\alpha + i^\star\alpha)$ the corresponding holomorphic differential in $\mathcal{H}(\Sigma)$. Let g be the metric on Σ given by the line element

$$ds_g = |\omega|. \tag{3.1}$$

In local coordinates, if $\omega = h(z)dz$, then

$$ds_g^2 = |h(z)dz|^2. \tag{3.2}$$

Thus g defines a flat metric on Σ with conical singularities at the $2h-2$ zeroes of ω. For an account on the geodesic dynamics of such surfaces see for example [3]. Let Δ_g be the corresponding Laplacian acting on degree zero bundles. In local coordinates

$$\Delta_g = -|h(z)|^{-2}\partial_z\partial_{\bar{z}}. \tag{3.3}$$

Theorem 1. If $k \in \mathbb{Z}$ and F_k is the single valued function

$$F_k(z) = e^{2\pi ik \int_{z_0}^{z} \alpha}, \tag{3.4}$$

then

$$\Delta_g F_k = k^2 F_k. \tag{3.5}$$

Proof. Immediate.

Let us define the single valued functions

$$h_{n,m} \doteq \frac{f_{n,m}}{\overline{f}_{n,m}} = e^{\int^z \omega_{n,m} - \overline{\int^z \omega_{n,m}}}. \tag{3.6}$$

Note that the functions F_k correspond to $h_{n,m}^k$ for some integer vectors n and m. For $k \in \mathbb{Z}\backslash\{0\}$ we have

$$\int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{n,m} \exp k \left(\int^z \omega_{n,m} - \overline{\int^z \omega_{n,m}} \right) = 0, \tag{3.7}$$

which follows from the fact that the integrand is a total derivative. Since $\overline{F}_k = F_k^{-1}$, it follows that the F_k satisfy the orthonormality relation

$$\int_\Sigma d\mu F_k \overline{F}_j = \delta_{jk}, \tag{3.8}$$

where

$$d\mu \doteq \frac{\omega \wedge \overline{\omega}}{\int_\Sigma \omega \wedge \overline{\omega}}. \tag{3.9}$$

3.1 Multivaluedness, area and eigenvalues

The area of Σ with respect to the metric $ds^2 = |\omega_{n,m}|^2$ is given by

$$A_{n,m} = \frac{i}{4} \int_\Sigma \omega_{n,m} \wedge \overline{\omega}_{n,m} = \frac{\pi}{2} \langle n, m | n, m \rangle = \frac{\pi^2}{2} (m - n \cdot \Omega) \cdot {\Omega^{(2)}}^{-1} \cdot (m - n \cdot \overline{\Omega}). \tag{3.10}$$

The multivaluedness of $f_{n,m}$ is related to $A_{n,m}$. In particular, winding around the cycle $\gamma_{n,-m} = -m \cdot \alpha + n \cdot \beta$, we have

$$\mathcal{P}_{n,-m} f_{n,m}(z) = e^{-\frac{2A_{n,m}}{\pi}} f_{n,m}(z) = e^{-\langle n,m|n,m \rangle} f_{n,m}(z), \tag{3.11}$$

where $\mathcal{P}_{q,p}$ is the winding operator

$$\mathcal{P}_{q,p} g(z) = g(z + \gamma_{p,q}). \tag{3.12}$$

Comparing (3.11) with (3.5) we get the following relationship connecting dilatations and eigenvalues

$$\lambda_k = -\pi \log \frac{\mathcal{P}^{k^2}_{n,-m} f_{n,m}(z)}{f_{n,m}(z)}. \tag{3.13}$$

Thus we can express the action of the Laplacian on $h^k_{n,m} = (f_{n,m}/\overline{f}_{n,m})^k$ in terms of the winding operator acting on $f_{n,m}$. This relationship between eigenvalues and multivaluedness is reminiscent of a similar relation arising between geodesic lengths (Fuchsian dilatations) and eigenvalues of the Poincaré Laplacian (Selberg trace formula). This is not a surprise. Actually, in the previous sections we have reproduced in the higher genus case some of the structures arising in the case of the torus. In particular, we considered the points $c_{n,m;k}$ for which the imaginary part of $\sum_k c_{n,m;k} \oint_{\gamma_{p,q}}$ takes values in $\pi\mathbb{Z}$. This is reminiscent of the Poisson summation formula where it (McKean).

3.2 Genus one

Let us denote by τ the Riemann period matrix in the case of the torus. We set $\tau^{(1)} \doteq \Re\,\tau$ and $\tau^{(2)} \doteq \Im\,\tau$. For $h = 1$ we have

$$\omega_{n,m} = c_{n,m}\omega, \tag{3.14}$$

with $\omega \doteq \omega_1$ the unique holomorphic one-differential on the torus such that $\oint_\alpha \omega = 1$. By (2.11) we have

$$c_{n,m} = \pi \frac{(m - n\overline{\tau})}{\tau^{(2)}}, \qquad (n,m) \in \mathbb{Z}^2. \tag{3.15}$$

The functions

$$h_{n,m} = e^{c_{n,m}\int^z \omega - \overline{c}_{n,m}\overline{\int^z \omega}}, \qquad (n,m) \in \mathbb{Z}^2, \tag{3.16}$$

coincide with the well-known eigenfunctions of the Laplacian $\Delta = -2\partial_z\partial_{\overline{z}}$

$$\Delta h_{n,m} = \lambda_{n,m} h_{n,m}, \qquad (n,m) \in \mathbb{Z}^2, \tag{3.17}$$

where

$$\lambda_{n,m} \doteq 2|c_{n,m}|^2 = 2\pi^2 \frac{(m - n\tau)(m - n\overline{\tau})}{{\tau^{(2)}}^2}. \tag{3.18}$$

Under modular transformations of τ the eigenvalues transform in the following way

$$\lambda_{n,m}(\tau + 1) = \lambda_{n,m-n}(\tau), \qquad \lambda_{n,m}\left(-\frac{1}{\tau}\right) = |\tau|^2 \lambda_{-m,n}(\tau). \tag{3.19}$$

The results in the previous sections shown that the eigenvalues can be generated by winding around the homology cycles, namely

$$\lambda_{n,m} = 2\pi \frac{\langle n,m|n,m\rangle}{\tau^{(2)}} = 2\pi \frac{\Re \oint_{\gamma_{m,-n}} \omega_{n,m}}{\tau^{(2)}}. \tag{3.20}$$

In other words, by acting with the winding operator we recover the full spectrum. We now prove modular invariance of ${\tau^{(2)}}^{-1}\det' \Delta$ without computing it (the prime indicates omission of the zero mode of Δ). First of all note that

$$\mu_{n,m}(\tau) = \tau^{(2)}\lambda_{n,m}(\tau), \quad (n,m) \in \mathbb{Z}^2, \tag{3.21}$$

which correspond to the eigenvalues of the Laplacian $\Delta' = \tau^{(2)}\Delta$, satisfy

$$\mu_{\gamma(n,m)}(\gamma \cdot \tau) = \mu_{n,m}(\tau), \tag{3.22}$$

where $\gamma(n,m) \doteq (\tilde{n}, \tilde{m})$ with

$$\begin{pmatrix} \tilde{m} \\ \tilde{n} \end{pmatrix} \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}, \tag{3.23}$$

and

$$\gamma \cdot \tau \doteq \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}). \tag{3.24}$$

By (3.22) we have $\mu_{\gamma(n,m)}(\tau) = \mu_{n,m}(\gamma^{-1} \cdot \tau)$, that is any $\mu_{n,m}$, and therefore $\lambda_{n,m}$, can be obtained from a given eigenvalue by modular transformations. The determinants of Δ and Δ' are related by

$$\det{}' \Delta' = \det{}' (\tau^{(2)}) \det{}' \Delta = {\tau^{(2)}}^{-1} \det (\tau^{(2)}) \det{}' \Delta, \tag{3.25}$$

where we used the fact that $c \det' c = \det c$, $c \in \mathbb{C}$. Since $\det \tau^{(2)}$ can be regularized by standard techniques, e.g. by the ζ-function regularization method, to a finite τ-independent constant, we have

$$\frac{\det' \Delta}{\tau^{(2)}} = const \prod_{(n,m)\in\mathbb{Z}^2\setminus\{0,0\}} \mu_{n,m} = const \prod_{(n,m)\in\mathbb{Z}^2\setminus\{0,0\}} \tau^{(2)}\lambda_{n,m}. \tag{3.26}$$

Modular invariance of ${\tau^{(2)}}^{-1}\det' \Delta$ now follows by (3.22), namely

$$\prod_{(n,m)\in\mathbb{Z}^2\setminus\{0,0\}} \tau^{(2)}\lambda_{n,m}(\tau) = \prod_{\gamma\in PSL(2,\mathbb{Z})} \mu_{N,M}(\gamma \cdot \tau), \tag{3.27}$$

where N, M are arbitrary integers not simultaneously vanishing. It follows that $\det' \Delta$ can be expressed as the product of all modular transformations acting on an arbitrary eigenvalue

$$\det' \Delta = const\, \tau^{(2)} \prod_{\gamma \in PSL(2,\mathbb{Z})} \mu_{N,M}(\gamma \cdot \tau). \tag{3.28}$$

Note that modular invariance of ${\tau^{(2)}}^{-1} \det' \Delta$ essentially implies that ${\tau^{(2)}}^{-1} \det' \Delta = \tau^{(2)} |\eta(\tau)|^4$, where $\eta(\tau)$ is the Dedekind η-function.

4 Special Riemann surfaces

In general there are other eigenfunctions besides $h^k_{n,m}$, $k \in \mathbb{N}$. For example, when all the $2m_j$'s and $2n_j$'s are integer multiple of an integer N then the eigenfunctions include $h^k_{n,m}$, $k \in \mathbb{N}$ whose eigenvalue is $2A_{n,m} k^2 / N^2$.

More generally one should investigate whether the period matrix has some non trivial number theoretical structure. To see this we first note the trivial fact that since for $h = 1$ the space of holomorphic one-differentials is one-dimensional, it follows that the ratio between $\omega_{n,m}$ and $\omega_{n',m'}$ is always a constant. This allows one to construct the infinite set of eigenvalues labelled by two integers $(n, m) \in \mathbb{Z}^2$. In the case $h \geq 2$ the ratio $\omega_{n,m}/\omega_{n',m'}$ is in general not a constant. This is the reason why we considered the eigenfunctions of the kind $h^k_{n,m}$ with fixed $(n, m) \in \mathbb{Z}^{2h}$. However there are other interesting possibilities. For example, if besides α also ${}^\star\alpha$ is integral, then for $k_1, k_2 \in \mathbb{Z}$, the single valued function

$$F_{k_1,k_2} = e^{2\pi i \int_{z_0}^{z} (k_1 \alpha + k_2 {}^\star\alpha)}, \tag{4.1}$$

satisfies

$$\Delta_g F_{k_1,k_2} = (k_1^2 + k_2^2) F_{k_1,k_2}. \tag{4.2}$$

Note that since $k_1 \alpha + k_2 {}^\star\alpha = 2i \Im\left[(k_1 - i k_2)\omega\right]$, it follows that integrality of both α and ${}^\star\alpha$ is a particular case of a more general one.

Theorem 2. If the holomorphic one-differentials $\omega_{n,m}$ and $\omega_{n',m'}$ satisfy the equation

$$\omega_{n',m'}(z) = c\, \omega_{n,m}(z), \tag{4.3}$$

with both (n, m) and (n', m') in $\mathbb{Z}^{2h}$ and $c \in \mathbb{C} \backslash \mathbb{Q}$, then the single valued function

$$h_{n',m'} = e^{c \int^z \omega_{n,m} - \overline{c} \overline{\int^z \omega_{n,m}}} \neq h^k_{n,m} \qquad c \in \mathbb{C} \backslash \mathbb{Q}, \quad k \in \mathbb{Q}, \tag{4.4}$$

satisfies

$$\Delta_{g^{n,m}} h_{n',m'} = \lambda_c h_{n',m'}, \tag{4.5}$$

where

$$\lambda_c = 2A_{n,m}|c|^2, \tag{4.6}$$

and

$$c = \frac{m'_i - \sum_{k=1}^h \overline{\Omega}_{ik} n'_k}{m_i - \sum_{k=1}^h \overline{\Omega}_{ik} n_k} = \frac{m'_j - \sum_{k=1}^h \overline{\Omega}_{jk} n'_k}{m_j - \sum_{k=1}^h \overline{\Omega}_{jk} n_k}, \quad i,j = 1,\ldots,h. \tag{4.7}$$

PROOF. It is immediate to check Eq.(4.5). The only point is to find the expression of c. This follows by the observation that since the holomorphic one-differentials $\omega_1,\ldots,\omega_h$ are linearly independent, Eq.(4.3) is equivalent to

$$m'_j - \sum_{k=1}^h \Omega_{jk} n'_k = \overline{c}\left(m_j - \sum_{k=1}^h \Omega_{jk} n_k\right), \qquad j = 1,\ldots,h. \tag{4.8}$$

Note that to each (n,m) and (n',m') satisfying (4.8) there is a possible value of $c \equiv c(n,m;n',m')$.

Thus we can reproduce in higher genus the basic structure Eqs.(3.16)-(3.18) considered in the torus case and then to find eigenvalues with a non trivial dependence on the complex structure.

The problem of finding the possible (in general complex) solutions of (4.3) is strictly related to the number theoretical properties of Ω.

Observe that

$$\Delta_{g^{n',m'}} h_{n,m} = {\lambda_c}' h_{n,m}, \tag{4.9}$$

where

$${\lambda_c}' = 4\frac{A_{n,m}A_{n',m'}}{\lambda_c}. \tag{4.10}$$

We will call *Special*, the Riemann surfaces admitting solutions of (4.8) with non rational values of c. Since a change in the sign of c is equivalent to a change sign of either (m,n) or (m',n'), in the following, without loss of generality, we will assume that

$$\Im\, \overline{c} > 0. \tag{4.11}$$

4.1 The solution space

Eq.(4.8) has solutions if there are integers (n, m) and (n', m') such that the period matrix satisfies the $h-1$ consistency conditions

$$\frac{m'_i - \sum_{k=1}^h \Omega_{ik} n'_k}{m_i - \sum_{k=1}^h \Omega_{ik} n_k} = \frac{m'_j - \sum_{k=1}^h \Omega_{jk} n'_k}{m_j - \sum_{k=1}^h \Omega_{jk} n_k}, \quad i, j = 1, \ldots, h. \tag{4.12}$$

Thus, for $(n, m) \in \mathbb{Z}^{2h}$ fixed, the Laplacian $\Delta_{g^{n,m}}$ has eigenvalues

$$\lambda_c = 2A_{n,m} \left| \frac{m'_i - \sum_{k=1}^h \overline{\Omega}_{ik} n'_k}{m_i - \sum_{k=1}^h \overline{\Omega}_{ik} n_k} \right|^2, \qquad (n', m') \in \mathcal{S}_{n,m}(\Omega), \tag{4.13}$$

where $\mathcal{S}_{n,m}(\Omega)$ denotes the *solution space*

$$\mathcal{S}_{n,m}(\Omega) \doteq \left\{ (n', m') \, \middle| \omega_{n',m'} = c\, \omega_{n,m}, (n', m') \in \mathbb{Z}^{2h} \right\}. \tag{4.14}$$

For a given Ω the space $\mathcal{S}_{n,m}(\Omega)$ may contain other points besides $(kn, km), k \in \mathbb{Z}$. To investigate the structure of such space we should understand the nature of the Riemann surfaces whose period matrix satisfies Eq.(4.8). To this end we note that Eq.(4.8) has been suggested in order to reproduce in higher genus the basic structure Eqs.(3.16)-(3.18) considered in the torus case. So, we should expect a Riemann surface strictly related to the torus geometry. Remarkably, this is in fact the case as we have the following

Theorem 3. The Riemann surfaces with period matrices satisfying Eq.(4.8) correspond to branched covering of the torus.

PROOF. First of all note that by (4.8) it follows that the function

$$w(z) = \int^z \hat{\omega}, \tag{4.15}$$

where $\hat{\omega} \doteq (\bar{c}n - n') \cdot \omega$, has monodromy

$$\oint_{\gamma_{p,q}} \hat{\omega} = -p \cdot n' - q \cdot m' + \bar{c}(p \cdot n + q \cdot m), \tag{4.16}$$

implying that w is a holomorphic map from Σ to the torus with period matrix $\bar{c}$. Viceversa, if w is a holomorphic map of a branched covering of the torus to the torus itself, then $w(z+\gamma_{p,q}) = w(z) + p \cdot N' + q \cdot M' + \tau(p \cdot N + q \cdot M)$, for some integer vectors M, N, M', N', and by the Riemann bilinear relations

$$0 = \int_\Sigma \omega_k \wedge dw = M'_k + \tau M_k - \sum_{j=1}^h (N'_j + \tau N_j)\Omega_{jk}, \qquad k = 1, \ldots, h, \tag{4.17}$$

which is Eq.(4.8) with $\tau = \bar{c}$ (see [4] for explicit constructions of branched covering of the torus[1]).

Remark 1. Eq.(1.2), derived in [1] (see Eqs.(5.18) and (5.23) there), has been subsequently and independently derived in [5] by studying the null compactification of type-IIA-string perturbation theory at finite temperature. In [5] an equivalent proof of **Theorem 3.** is also provided.

4.2 Metric induced by the covering

By means of the map w from Σ to the torus with period matrix $sign\,(\Im\,(\bar{c}))\bar{c}$ we can construct an infinite set of eigenfunctions for the Laplacian $\Delta = -2|\hat{\omega}|^{-2}\partial_z\partial_{\bar{z}}$ on Σ defined with respect to the metric $|\hat{\omega}|^2$. These eigenfunctions are

$$h_{n,m} = e^{c_{n,m}\int^z \hat{\omega} - \bar{c}_{n,m}\overline{\int^z \hat{\omega}}}, \qquad (n,m) \in \mathbb{Z}^2, \tag{4.18}$$

corresponding to the eigenvalues

$$\lambda_{n,m} = 2|c_{n,m}|^2 = 2\pi^2 \frac{(m-cn)(m-\bar{c}n)}{{c^{(2)}}^2}, \tag{4.19}$$

where

$$c_{n,m} = \pi\frac{(m-n\bar{c})}{c^{(2)}}, \qquad (n,m) \in \mathbb{Z}^2. \tag{4.20}$$

The proof is a direct consequence of the relation $\Delta = -2|\hat{\omega}|^{-2}\partial_z\partial_{\bar{z}} = -2\partial_w\partial_{\bar{w}}$.

Let us set

$$D_j \doteq \sum_{k=1}^h \overline{D}_{kj}^{mn}, \qquad D'_j \doteq \sum_{k=1}^h \overline{D}_{kj}^{m'n'}. \tag{4.21}$$

By Eq.(4.7) and the Poisson summation formula, we have

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi\frac{D'_j}{D_j}} = \sqrt{\frac{D_j}{D'_j}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi\frac{D_j}{D'_j}}, \qquad j = 1, \ldots, h. \tag{4.22}$$

4.3 Genus 2

Before considering Eq.(4.12) for arbitrary genus, it is instructive to investigate the $h = 2$ case. Since $\Omega_{ij} = \Omega_{ji}$ we have

$$\frac{m'_1 - \Omega_{11}n'_1 - \Omega_{12}n'_2}{m_1 - \Omega_{11}n_1 - \Omega_{12}n_2} = \frac{m'_2 - \Omega_{12}n'_1 - \Omega_{22}n'_2}{m_2 - \Omega_{12}n_1 - \Omega_{22}n_2}. \tag{4.23}$$

[1] I am grateful to the anonymous referee for suggesting Ref.[4].

Thus the problem is the following: given Ω_{11} and Ω_{12} find all the integers $(n, m; n', m') \in \mathbb{Z}^8$ such that (4.23) is satisfied. The solution of this problem depends on structure of Ω.

We consider period matrices satisfying the relation

$$\Omega_{22} = \frac{\hat{N}_1}{\hat{N}_4}\Omega_{11} + \frac{\hat{N}_2}{\hat{N}_4}\Omega_{12} + \frac{\hat{N}_3}{\hat{N}_4}, \qquad (\hat{N}_1, \hat{N}_2, \hat{N}_3, \hat{N}_4) \in \mathbb{Z}^4, \qquad \hat{N}_4 \neq 0, \tag{4.24}$$

that is

$$\Omega_{ij} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & N_1\Omega_{11} + N_2\Omega_{12} + N_3 \end{pmatrix}, \tag{4.25}$$

where $N_i \doteq \hat{N}_i/\hat{N}_4$, $i = 1, 2, 3$. Positivity of $\Omega^{(2)}_{ij}$ implies the following condition on N_1, N_2, N_3

$$\Omega^{(2)}_{11}\left(N_1\Omega^{(2)}_{11} + N_2\Omega^{(2)}_{12} + N_3\right) > {\Omega^{(2)}_{12}}^2. \tag{4.26}$$

With the position (4.24) Eq.(4.23) becomes

$$\frac{m'_1 - \Omega_{11}n'_1 - \Omega_{12}n'_2}{m_1 - \Omega_{11}n_1 - \Omega_{12}n_2} = \frac{m'_2 - N_3n'_2 - \Omega_{11}N_1n'_2 - \Omega_{12}(n'_1 + N_2n'_2)}{m_2 - N_3n_2 - \Omega_{11}N_1n_2 - \Omega_{12}(n_1 + N_2n_2)}. \tag{4.27}$$

We will look for solutions of this equation of the form

$$m_2 - N_3n_2 - \Omega_{11}N_1n_2 - \Omega_{12}(n_1 + N_2n_2) = N\left[m_1 - \Omega_{11}n_1 - \Omega_{12}n_2\right], \tag{4.28}$$

$$m'_2 - N_3n'_2 - \Omega_{11}N_1n'_2 - \Omega_{12}(n'_1 + N_2n'_2) = N\left[m'_1 - \Omega_{11}n'_1 - \Omega_{12}n'_2\right], \tag{4.29}$$

with $N \in \mathbb{Q}$. Since $(n, m; n', m') \in \mathbb{Z}^8$, the general solutions not depending on Ω_{11} and Ω_{12} are

$$m_2 - N_3n_2 = Nm_1, \quad n_1 + N_2n_2 = Nn_2, \quad N_1n_2 = Nn_1, \tag{4.30}$$

$$m'_2 - N_3n'_2 = Nm'_1, \quad n'_1 + N_2n'_2 = Nn'_2, \quad N_1n'_2 = Nn'_1. \tag{4.31}$$

Note that each solution of Eq.(4.30) defines a metric $g^{n,m}$ whereas the solutions of Eq.(4.31) give, by (4.13), the eigenvalues λ_c of $\Delta_{g^{n,m}}$.

The compatibility condition for (4.31) constrains N to be

$$N_\pm = \frac{N_2 \pm \sqrt{N_2^2 + 4N_1}}{2}. \tag{4.32}$$

Since $N \in \mathbb{Q}$ and $N_i \in \mathbb{Z}/\hat{N}_4$, $i = 1, 2, 3$, we have

$$N_1 = MN_2 + M^2, \qquad M \in \left\{k \in \mathbb{Q} \,\middle|\, (kN_2 + k^2) \in \mathbb{Z}/\hat{N}_4\right\}. \tag{4.33}$$

Eq.(4.30) has a double set of solutions. In the case $N = N_+ = N_2 + M$, we have

$$n_2 = \frac{n_1}{M}, \qquad m_2 = (N_2 + M)m_1 + \frac{N_3 n_1}{M}, \qquad (n_1, m_1) \in \Gamma^{(+)}, \tag{4.34}$$

where

$$\Gamma^{(+)} \doteq \left\{ (k, j) \in \mathbb{Z}^2 \middle| \left(\frac{k}{M}, (N_2 + M)j + \frac{N_3 k}{M} \right) \in \mathbb{Z}^2 \right\}. \tag{4.35}$$

In the second case $N = N_- = -M$, so that

$$n_2 = -\frac{n_1}{N_2 + M}, \qquad m_2 = -Mm_1 - \frac{N_3 n_1}{N_2 + M}, \qquad (n_1, m_1) \in \Gamma^{(-)}, \tag{4.36}$$

where

$$\Gamma^{(-)} \doteq \left\{ (k, j) \in \mathbb{Z}^2 \middle| \left(\frac{k}{N_2 + M}, Mj + \frac{N_3 k}{N_2 + M} \right) \in \mathbb{Z}^2 \right\}. \tag{4.37}$$

Note that given N_1, N_2 and N_3 we found that $N_1 = MN_2 + M^2$ and either $N = N_2 + M$ or $-M$. Therefore it is natural to choose M, N_2 and N_3 to parametrize Ω_{22}. Note also that by (4.24) and (4.33) it follows that M and $-N_2 - M$ correspond to the same value of Ω_{22}.

Given M, N_2, N_3, N_4 there are two sets of Laplacians parametrized by points in $\Gamma^{(\pm)}$. Let (n_1, m_1) be a point in $\Gamma^{(+)}$ and (n_2, m_2) given by (4.34). The first set of eigenvalues is

$$\lambda_c^{(+)} = 2A_{n,m} \frac{|Mm'_1 - n'_1 (M\Omega_{11} + \Omega_{12})|^2}{|Mm_1 - n_1 (M\Omega_{11} + \Omega_{12})|^2}, \tag{4.38}$$

where

$$n'_2 = \frac{n'_1}{M}, \quad m'_2 = (N_2 + M)m'_1 + \frac{N_3 n'_1}{M}, \qquad (n'_1, m'_1) \in \Gamma^{(+)}. \tag{4.39}$$

Let (n_1, m_1) be a point in $\Gamma^{(-)}$ and (n_2, m_2) given by (4.36). The second set of eigenvalues is

$$\lambda_c^{(-)} = 2A_{n,m} \frac{|(N_2 + M)\, m'_1 - n'_1 \left[(N_2 + M)\, \Omega_{11} - \Omega_{12}\right]|^2}{|(N_2 + M)m_1 - n_1 \left[(N_2 + M)\, \Omega_{11} - \Omega_{12}\right]|^2}, \tag{4.40}$$

where

$$n'_2 = -\frac{n'_1}{N_2 + M}, \quad m'_2 = -Mm_1 - \frac{N_3 n_1}{N_2 + M}, \qquad (n'_1, m'_1) \in \Gamma^{(-)}. \tag{4.41}$$

These eigenvalues have a structure which is similar to the structure of the ones of the Laplacian on the torus.

4.4 Higher genus

We now consider the higher genus case. By (4.12) we have

$$m_i - \sum_{k=1}^{h} \Omega_{ik} n_k = N_{ij} \left(m_j - \sum_{k=1}^{h} \Omega_{jk} n_k \right), \tag{4.42}$$

$$m_i' - \sum_{k=1}^{h} \Omega_{ik} n_k' = N_{ij} \left(m_j' - \sum_{k=1}^{h} \Omega_{jk} n_k' \right) . \tag{4.43}$$

$i, j = 1, \ldots, h$. Eq.(4.42) gives the constraint on the structure of the metric $g^{n,m}$. Note that

$$N_{ij} N_{jk} = N_{ik}, \quad i, j = 1, \ldots, h, \tag{4.44}$$

in particular, $N_{ij} = N_{ji}^{-1}$. The matrix N_{ij} is determined by $h-1$ elements. For example, since $N_{ij} = N_{i1} N_{1j} = N_{1i}^{-1} N_{1j}$, in terms of $N_{12}, \ldots, N_{1h}$ we have

$$N_{ij} = \begin{pmatrix} 1 & N_{12} & N_{13} & \ldots & N_{1h} \\ N_{12}^{-1} & 1 & N_{13} N_{12}^{-1} & \ldots & N_{1h} N_{12}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ N_{1h}^{-1} & N_{1h}^{-1} N_{12} & N_{1h}^{-1} N_{13} & \ldots & 1 \end{pmatrix} , \tag{4.45}$$

which has vanishing determinant.

Since Ω is symmetric, it follows that in each one of the $h-1$ equations Eq.(4.42) (or Eq.(4.43)) there is always one, and only one, matrix element appearing in both sides. In other words, both sides of Eq.(4.42) contain $\Omega_{ij} = \Omega_{ji}$. Therefore, it is natural to consider period matrices of the form

$$\Omega_{ij} = \sum_{k,l=1}^{h} N_{ij}^{kl} \Omega_{kl} + M_{ij}, \qquad N_{ij}^{kl} \in \mathbb{Q}, \quad M_{ij} \in \mathbb{Q}, \qquad N_{ij}^{ij} = 0, \tag{4.46}$$

and then, for each pair i, j, to substitute it in the equation involving N_{ij}. This allows us to transform, for each i, j, Eqs.(4.42)(4.43) in equations containing all the matrices elements but Ω_{ij} on both sides. Note that the symmetry of Ω_{ij} implies that N_{ij}^{kl} is symmetric in i, j and k, l separately and $M_{ij} = M_{ji}$. By (4.46) we have that Eqs.(4.42)(4.43) become

$$N_{ij} m_j - m_i + \sum_{k,l=1}^{h} \left\{ (n_j - n_i N_{ij}) \left(N_{ij}^{kl} \Omega_{kl} + M_{ij} \right) + \Omega_{kl} \left[\delta_{ik} (n_l - \delta_{lj} n_j) - \delta_{jk} (n_l - \delta_{li} n_i) N_{ij} \right] \right\} = 0, \tag{4.47}$$

$$N_{ij} m_j' - m_i' + \sum_{k,l=1}^{h} \left\{ (n_j' - n_i' N_{ij}) \left(N_{ij}^{kl} \Omega_{kl} + M_{ij} \right) + \Omega_{kl} \left[\delta_{ik} (n_l' - \delta_{lj} n_j') - \delta_{jk} (n_l' - \delta_{li} n_i') N_{ij} \right] \right\} = 0, \tag{4.48}$$

$i, j = 1, \ldots, h$. We do not investigate the conditions following from Eqs.(4.42)(4.43) further, rather we shortly consider the period matrices satisfying the conditions

$$\Omega_{ik} = \sum_{l=1}^{h} N_{ik,j}^{l} \Omega_{jl} + M_{ik}, \qquad N_{ik,j}^{l} \in \mathbb{Q}, \ M_{ik} \in \mathbb{Q}, \quad i, j, k = 1, \ldots, h. \tag{4.49}$$

Substituting Ω_{ik} in the left hand side of (4.42)(4.43) these transform in simplified equations as now they involve matrix elements with the same value of the first index.

Substituting Ω_{jl} in the RHS of (4.49) with $\sum_{m=1}^{h} N_{jl,n}^{m}\Omega_{nm} + M_{jl}$, we have

$$\Omega_{ik} = \sum_{l=1}^{h} N_{ik,j}^{l} \left(\sum_{m=1}^{h} N_{jl,n}^{m}\Omega_{nm} + M_{jl} \right) + M_{ik}, \quad i,j,k,n = 1,\ldots,h. \tag{4.50}$$

Comparing (4.49) with (4.50) one obtains a set of equations that, once one makes the additional requirement that the terms involving the period matrix cancel separately, become

$$\sum_{l=1}^{h} N_{ik,j}^{l} N_{jl,n}^{m} = N_{ik,n}^{m}, \quad i,j,k,m,n = 1,\ldots,h, \tag{4.51}$$

$$\sum_{l=1}^{h} N_{ik,j}^{l} M_{jl} = 0, \quad i,j,k = 1,\ldots,h. \tag{4.52}$$

4.5 Special Riemann surfaces and complex multiplication

We now show that Special Riemann surfaces have a Jacobian with Complex Multiplication (CM).[2] First note that in terms of

$$v_k \doteq m_k - \sum_{j=1}^{h} \Omega_{kj} n_j, \qquad v'_k \doteq m'_k - \sum_{j=1}^{h} \Omega_{kj} n'_j, \tag{4.53}$$

Eq.(4.12) reads

$$v_j v'_k - v_k v'_j = 0, \qquad \forall j,k, \tag{4.54}$$

which is equivalent to

$$\Omega N \Omega + \Omega M - \tilde{M}\Omega - M' = 0, \tag{4.55}$$

where ˜ denotes the transpose and

$$N_{jk} \doteq n'_j n_k - n'_k n_j, \qquad M_{jk} \doteq m'_j n_k - m_k n' j, \qquad M'_{jk} \doteq m'_j m_k - m'_k m_j. \tag{4.56}$$

On the other hand, a Jacobian is said to admit complex multiplication if there exist integer matrices N, M, M' and N' such that

$$\Omega(M + N\Omega) = M' + N'\Omega, \tag{4.57}$$

that is

$$\Omega N \Omega + \Omega M - N'\Omega - M' = 0. \tag{4.58}$$

[2]Jacobian with CM have been recently considered in the framework of rational CFT [6].

Comparing with (4.55), we see that the Jacobians of Special Riemann surfaces admit a particular kind of CM. According to Theorem 3. this CM is the one of Jacobian corresponding to branched covering of the torus.

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